N-fold Darboux transformation and soliton solutions for a nonlinear Dirac system

2005 J. Phys. A: Math. Gen. 38 1063

(http://iopscience.iop.org/0305-4470/38/5/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 1063–1069 doi:10.1088/0305-4470/38/5/008

N-fold Darboux transformation and soliton solutions for a nonlinear Dirac system

E G Fan

Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

E-mail: faneg@fudan.edu.cn

Received 8 November 2004
Published 19 January 2005
Online at stacks.iop.org/JPhysA/38/1063

Abstract
From a spectral problem and corresponding Lenard operator pairs, we derive a Dirac soliton hierarchy associated with a nonlinear Dirac system. A systematic method is proposed for constructing the N-fold Darboux transformation of the Dirac system based on its Lax pair. As an application of Darboux transformation, explicit soliton solutions of the Dirac system are given.

PACS number: 05.45.Yv

1. Introduction

The investigation of the exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modelled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The exact solution, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. In the past decades, there has been significant progress in the development of various methods. Among them, Darboux transformation is a powerful method to get exact solutions of nonlinear partial differential equations. The key for constructing Darboux transformation is to expose kinds of covariant properties that the corresponding spectral problems possess. There have been many tricks to do this for getting explicit solutions to various soliton equations including the KdV equation, KP equation, Davey–Stewartson equation, Yang–Mills flows, etc [1–8].

We consider the Dirac spectral problem

$$\psi_x = U\psi = \begin{pmatrix} q & \lambda + r \\ -\lambda + r & -q \end{pmatrix} \psi, \tag{1.1}$$

which was first introduced by Frolov [9]. Time evolution of the scattering data of problem (1.1) was discussed by Grosse [10] and a detailed analysis of the inverse scattering problem was provided by Hinton *et al* [11] for a more general spectral problem. Binary nonlinearization and Dirac soliton hierarchy associated with problem (1.1) were fully studied by Ma [12]. A hierarchy of first-degree time-dependent symmetries for the Dirac hierarchy was further constructed [13]. The first nonlinear Dirac system in the Dirac soliton hierarchy is as follows [12, 13]:

$$q_t = \tfrac{1}{2}r_{xx} - q^2 r - r^3, \qquad r_t = -\tfrac{1}{2}q_{xx} + qr^2 + q^3. \tag{1.2}$$

By gauge transformation

$$\tilde{\psi}_x = \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \psi, \qquad u = q - \mathrm{i}r \quad v = q + \mathrm{i}r, \tag{1.3}$$

and simple calculation, we know that the Dirac spectral problem (1.1) is equivalent to the standard AKNS spectral problem

$$\tilde{\psi}_x = \begin{pmatrix} \mathrm{i}\lambda & u \\ v & -\mathrm{i}\lambda \end{pmatrix} \tilde{\psi}. \tag{1.4}$$

Also the potentials q and r in the Dirac system are just the real and imaginary parts of the potentials u and v of the AKSN system. So the Dirac system (1.2) is similar to the coupled nonlinear Schrödinger system in the AKNS hierarchy. In fact, system (1.2) is exactly a member of the AKNS-D hierarchy [14]. In this paper, we are interested in the Darboux transformation and exact solutions of system (1.2) which is still unknown to our knowledge. In section 2, we derive from problem (1.1) the Dirac soliton hierarchy by using Lenard operator pairs which is silently different from Ma's method. The Lax pair for system (1.1) is further obtained. In section 3, a systematic method is proposed for constructing the N-fold Darboux transformation for the Dirac system (1.2). In section 4, we show an application of the Darboux transformation obtained. Soliton solutions of system (1.2) are given by applying its Darboux transformation. A short conclusion will be given in section 5.

2. The Dirac hierarchy and Lax pairs

To derive the Dirac hierarchy, we introduce the Lenard gradient sequence $\{S_j,\ j = 0, 1, \ldots\}$ by

$$KS_{j-1} = JS_j, \qquad S_j|_{(q,r)=(0,0)} = 0, \qquad S_0 = (0, 0, -1)^T \tag{2.1}$$

where $S_j = \left(S_j^{(1)}, S_j^{(2)}, S_j^{(3)}\right)^T$ and

$$K = \begin{pmatrix} \partial & 0 & 2r \\ 0 & \partial & -2q \\ 2r & -2q & \partial \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2r & -2q & \partial \end{pmatrix}. \tag{2.2}$$

Here and in the following context, we denote $\partial = \frac{\partial}{\partial x}$. It is easy to see that (2.1) and (2.2) imply the relation

$$2rS_j^{(1)} - 2qS_j^{(2)} + S_{jx}^{(3)} = 0, \tag{2.3}$$

and S_j is uniquely determined by the recursion relation (2.1). Here the condition $S_j|_{(q,r)=(0,0)} = 0$ is used to select the integration constant to be zero. A direct calculation gives

$$S_1 = \begin{pmatrix} -q \\ -r \\ 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} \frac{1}{2}r_x \\ -\frac{1}{2}q_x \\ -\frac{1}{2}(q^2 + r^2) \end{pmatrix}, \qquad S_3 = \begin{pmatrix} \frac{1}{4}(q_{xx} - 2q^3 - 2qr^2) \\ \frac{1}{4}(r_{xx} - 2r^3 - 2q^2 r) \\ \frac{1}{2}(qr_x - rq_x) \end{pmatrix}, \quad \cdots. \tag{2.4}$$

Consider the auxiliary problem

$$\psi_t = V^{(m)}\psi = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix}\psi, \tag{2.5}$$

where

$$V_{11}^{(m)} = \sum_{j=0}^{m} S_j^{(1)}\lambda^{m-j}, \qquad V_{12}^{(m)} = \sum_{j=0}^{m}\left(S_j^{(2)} + S_j^{(3)}\right)\lambda^{m-j},$$

$$V_{21}^{(m)} = \sum_{j=0}^{m}\left(S_j^{(2)} - S_j^{(3)}\right)\lambda^{m-j}.$$

Then the compatibility condition between (1.1) and (2.5) yields the zero-curvature equation $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to the following hierarchy of equations:

$$q_{t_m} = -2S_{m+1}^{(2)}, \qquad r_{t_m} = 2S_{m+1}^{(1)}.$$

The hierarchy can also be written as

$$\left(q_{t_m}, r_{t_m}\right)^T = X_m, \qquad m = 1, 2, \ldots \tag{2.6}$$

and

$$X_m = \begin{pmatrix} -2S_{m+1}^{(2)} \\ 2S_{m+1}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} S_{m+1}^{(1)} \\ S_{m+1}^{(2)} \end{pmatrix}.$$

The second member in the hierarchy (2.6) with $m = 2(X_2$-flow) is exactly the nonlinear Dirac system (1.2). From (2.4) and (2.5), we find that the Lax pair for system (1.2) consists of the spectral problem (1.1) and the following auxiliary problem:

$$\psi_t = V^{(2)}\psi = \begin{pmatrix} \frac{1}{2}r_x - q\lambda & -\frac{1}{2}q_x - \frac{1}{2}(q^2 + r^2) - r\lambda - \lambda^2 \\ -\frac{1}{2}q_x + \frac{1}{2}(q^2 + r^2) - r\lambda + \lambda^2 & -\frac{1}{2}r_x + q\lambda \end{pmatrix}\psi. \tag{2.7}$$

The Lax pairs (1.1) and (2.7) shall play a key role in the construction of the Darboux transformation for system (1.2).

3. Darboux transformation

In this section, we shall construct an *N*-fold Darboux transformation for the Dirac system (1.2). The Darboux transformation is actually a special gauge transformation

$$\tilde{\psi} = T\psi \tag{3.1}$$

of the solutions of the Lax pairs (1.1) and (2.7). It is required that $\tilde{\psi}$ also satisfies the Lax pairs (1.1) and (2.7) with some $\tilde{U}$ and $\tilde{V}^{(2)}$, i.e.

$$\tilde{\psi}_x = \tilde{U}\tilde{\psi}, \qquad \tilde{U} = (T_x + TU)T^{-1}, \tag{3.2}$$

$$\tilde{\psi}_t = \tilde{V}^{(2)}\tilde{\psi}, \qquad \tilde{V}^{(2)} = (T_t + TV^{(2)})T^{-1}. \tag{3.3}$$

By cross differentiating (3.2) and (3.3), we get

$$\tilde{U}_t - \tilde{V}_x^{(2)} + [\tilde{U}, \tilde{V}^{(2)}] = T\left(U_t - V_x^{(2)} + [U, V^{(2)}]\right)T^{-1}, \tag{3.4}$$

which implies that in order to make system (1.2) invariant under the gauge transformation (3.1), we should require that $\tilde{U}$ and $\tilde{V}^{(2)}$ have the same forms as U and $V^{(2)}$ respectively. At the same time the old potentials q and r in U, $V^{(2)}$ will be mapped into new potentials $\tilde{q}$

and $\tilde{r}$ in $\tilde{U}$, $\tilde{V}^{(2)}$. This process can be done continually and usually it may yield a series of multi-soliton solutions. We can construct the N-fold Darboux transformation for the Dirac system (1.2) as follows.

Let $(\phi_1(x,t,\lambda), \phi_2(x,t,\lambda))^T$ and $(\psi_1(x,t,\lambda), \psi_2(x,t,\lambda))^T$ be two basic solutions of the spectral problem (1.1) and (2.7), and use them to define two linear algebraic systems for A_k, B_k, C_k and D_k $(0 \leqslant k \leqslant N-1)$

$$\sum_{k=0}^{N-1}(A_k + D_k + \alpha_j B_k)\lambda_j^k = -\lambda_j^N, \qquad 1 \leqslant j \leqslant 2N, \tag{3.5}$$

$$\sum_{k=0}^{N-1}(C_k + \alpha_j A_k - \alpha_j D_k)\lambda_j^k = -\alpha_j\lambda_j^N, \qquad 1 \leqslant j \leqslant 2N \tag{3.6}$$

with

$$\alpha_j = \frac{\phi_2(\lambda_j) - \gamma_j\psi_2(\lambda_j)}{\phi_1(\lambda_j) - \gamma_j\psi_1(\lambda_j)}, \qquad 1 \leqslant j \leqslant 2N, \tag{3.7}$$

where λ_j and γ_j are some parameters suitably chosen such that determinants of the coefficients for (3.5) and (3.6) are nonzero. Hence, A_k, B_k, C_k and D_k are uniquely determined by (3.5) and (3.6). Now we let

$$T = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = I\lambda^N + \sum_{k=1}^{N-1} Q_k\lambda^k, \qquad Q_k = \begin{pmatrix} A_k + D_k & B_k \\ C_k & A_k - D_k \end{pmatrix}, \tag{3.8}$$

which is an Nth-order polynomial in λ with matrix coefficient. From (3.5), (3.6) and (3.8), it is easy to see that $\det T(\lambda)$ is the $2N$th-order polynomial of λ, and λ_j $(1 \leqslant j \leqslant 2N)$ are all its roots. Therefore, we have

$$\det T(\lambda) = \prod_{j=1}^{2N}(\lambda - \lambda_j). \tag{3.9}$$

Proposition 1. *The matrix $\tilde{U}$ determined by (3.2) has the same form as U, that is,*

$$\tilde{U} = \begin{pmatrix} \tilde{q} & \lambda + \tilde{r} \\ -\lambda + \tilde{r} & -\tilde{q} \end{pmatrix},$$

where the transformations between q, r and $\tilde{q}, \tilde{r}$ are given by

$$\tilde{q} = q - B_{N-1} - C_{N-1}, \qquad \tilde{r} = r + 2D_{N-1}. \tag{3.10}$$

The transformation $(\psi, q, r) \to (\tilde{\psi}, \tilde{q}, \tilde{r})$ is called a Darboux transformation of the spectral problem (1.1).

Proof. Let $T^{-1} = T^*/\det T$ and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}. \tag{3.11}$$

It is easy to see that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are $2N$th-order polynomials in λ, and $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are $(2N+1)$th-order polynomials in λ.

On the other hand, making use of (1.1), (3.5)–(3.7), we find that

$$\alpha_{jx} = -\lambda_j + r - 2q\alpha_j - (\lambda_j + r)\alpha_j^2,$$
$$A(\lambda_j) = -\alpha_j B(\lambda_j), \qquad C(\lambda_j) = -\alpha_j D(\lambda_j).$$

From the above equalities, it is easy to verify that all λ_j $(1 \leqslant j \leqslant 2N)$ are roots of $f_{kj}(\lambda)$ $(k, j = 1, 2)$, which together with (3.9) implies that $f_{kj}(\lambda)$ may be divided by $\det T$, and thus $(T_x + TU)T^{-1}$ is a first-order polynomial in λ with matrix coefficients, that is

$$T_x + TU = (\tilde{U}_1\lambda + \tilde{U}_0)T, \tag{3.12}$$

where the matrices $\tilde{U}_1(x, t)$ and $\tilde{U}_0(x, t)$ do not depend on λ.

We denote $U = U_1\lambda + U_0$, with

$$U_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} q & r \\ r & -q \end{pmatrix}.$$

Comparing the coefficients of λ^{N+1} and λ^N in (3.12) yields the following:

- $(N + 1)$th coefficient:

$$\tilde{U}_1 = U_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

- Nth coefficient:

$$\begin{aligned} \tilde{U}_0 &= U_0 + Q_{N-1}U_1 - \tilde{U}_1 Q_{N-1} \\ &= \begin{pmatrix} q - B_{N-1} - C_{N-1} & r + 2D_{N-1} \\ r + 2D_{N-1} & -q + B_{N-1} + C_{N-1} \end{pmatrix} = \begin{pmatrix} \tilde{q} & \tilde{r} \\ \tilde{r} & -\tilde{q} \end{pmatrix}, \end{aligned}$$

where $\tilde{q}$ and $\tilde{r}$ are given by (3.10).

The proof is completed. □

Next, we try to prove that $\tilde{V}^{(2)}$ in (3.3) has the same form as $V^{(2)}$ under the transformations (3.1) and (3.10).

Proposition 2. *The matrix $\tilde{V}^{(2)}$ in (3.3) has the same form as $V^{(2)}$ under the same transformations (3.1) and (3.10).*

Proof. In a way similar to proposition 1, we can prove that $(T_t + TV^{(2)})T^{-1}$ is a second-order polynomial in λ with matrix coefficients, that is

$$T_t + TV^{(2)} = (\tilde{V}_2\lambda^2 + \tilde{V}_1\lambda + \tilde{V}_0)T. \tag{3.13}$$

We write $V^{(2)}$ in the form $V^{(2)} = V_2\lambda^2 + V_1\lambda + V_0$, with

$$V_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} -q & -r \\ -r & q \end{pmatrix}$$

$$V_0 = \begin{pmatrix} \frac{1}{2}r_x & -\frac{1}{2}q_x - \frac{1}{2}(q^2 + r^2) \\ -\frac{1}{2}q_x + \frac{1}{2}(q^2 + r^2) & -\frac{1}{2}r_x \end{pmatrix}.$$

Comparing the coefficients of λ^{N+j} $(j = 0, 1, 2)$ yields the following:

- $(N + 2)$th coefficient:

$$\tilde{V}_2 = V_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- $(N + 1)$th coefficient:

$$\begin{aligned} \tilde{V}_1 &= V_1 + Q_{N-1}V_2 - \tilde{V}_2 Q_{N-1} \\ &= \begin{pmatrix} -q + B_{N-1} + C_{N-1} & -r - 2D_{N-1} \\ -r - 2D_{N-1} & q - B_{N-1} - C_{N-1} \end{pmatrix} = \begin{pmatrix} -\tilde{q} & -\tilde{r} \\ -\tilde{r} & \tilde{q} \end{pmatrix}. \end{aligned}$$

- Nth coefficient:

$$\tilde{V}_0 = V_0 + [Q_{N-1}, V_1] + Q_{N-2}V_2 - \tilde{V}_2 Q_{N-2}. \tag{3.14}$$

Again comparing the coefficient of λ^{N-1} in (3.12), we find that

$$Q_{N-1x} + [Q_{N-1}, U_0] + Q_{N-2}U_1 - \tilde{U}_1 Q_{N-2} = 0. \tag{3.15}$$

From (3.10) implies an identity

$$2(\tilde{q} - q)D_{N-1} + (\tilde{r} - r)(B_{N-1} + C_{N-1}) = 0. \tag{3.16}$$

By using (3.15) and (3.16), direct calculation shows that $\tilde{V}_0$ possess the same form as V_0. The proof is completed. □

Propositions 1 and 2 show that the transformations (3.1) and (3.10) change the Lax pairs (1.1) and (2.7) into another Lax pairs (3.2) and (3.3) in the same type. Therefore both the Lax pairs lead to the same Dirac system (1.2). We call the transformation $(\psi, q, r) \to (\tilde{\psi}, \tilde{q}, \tilde{r})$ a Darboux transformation of the Dirac system (1.2). In summary, we arrive at

Theorem 1. *The solutions (q, r) of the nonlinear Dirac system (1.2) are mapped into their new solution $(\tilde{q}, \tilde{r})$ under Darboux transformations (3.1) and (3.10), where B_{N-1}, C_{N-1} and D_{N-1} are given by (3.5) and (3.6).*

4. Application of the Darboux transformation

In this section, we shall apply the Darboux transformation (3.10) to construct explicit solutions of the nonlinear Dirac system (1.2). As usual we make a Darboux transformation starting from a special solution of system (1.2). Substituting $q = r = 0$ into the Lax pairs (1.1) and (2.7), we find that two basic solutions can be chosen as

$$\phi(\lambda) = \begin{pmatrix} \sin(\lambda x - \lambda^2 t) \\ \cos(\lambda x - \lambda^2 t) \end{pmatrix}, \qquad \psi(\lambda) = \begin{pmatrix} -\cos(\lambda x - \lambda^2 t) \\ \sin(\lambda x - \lambda^2 t) \end{pmatrix}.$$

According to (3.7), we have

$$\alpha_j = \frac{\cos\xi_j - \gamma_j \sin\xi_j}{\sin\xi_j + \gamma_j \cos\xi_j}, \qquad 1 \leqslant j \leqslant 2N, \tag{4.1}$$

where γ_j are constants, and $\xi_j = \lambda_j x - \lambda_j^2 t$. We shall discuss only the case $N = 1$.

Solving the linear algebraic system (3.5) and (3.6) yields

$$2D_0 = \frac{(\lambda_1 - \lambda_2)(\alpha_1 + \alpha_2)}{\alpha_1 - \alpha_2}, \qquad B_0 + C_0 = \frac{(\lambda_1 - \lambda_2)(\alpha_1\alpha_2 - 1)}{\alpha_1 - \alpha_2}. \tag{4.2}$$

Substituting (4.1) and (4.2) into the Darboux transformation (3.10), we then obtain a kind of soliton solutions for the Dirac system (1.2),

$$\begin{aligned} \tilde{q} &= \frac{(\lambda_2 - \lambda_1)[(1 - \gamma_1\gamma_2)\cos(\xi_1 + \xi_2) - (\gamma_1 + \gamma_2)\sin(\xi_1 + \xi_2)]}{(1 + \gamma_1\gamma_2)\sin(\xi_2 - \xi_1) + (\gamma_2 - \gamma_1)\cos(\xi_2 - \xi_1)}, \\ \tilde{r} &= \frac{(\lambda_1 - \lambda_2)[(1 - \gamma_1\gamma_2)\sin(\xi_1 + \xi_2) + (\gamma_1 + \gamma_2)\cos(\xi_1 + \xi_2)]}{(1 + \gamma_1\gamma_2)\sin(\xi_2 - \xi_1) + (\gamma_2 - \gamma_1)\cos(\xi_2 - \xi_1)}, \end{aligned} \tag{4.3}$$

where $\xi_1 = \lambda_1 x - \lambda_1^2 t$, $\xi_2 = \lambda_2 x - \lambda_2^2 t$. The plots for solutions $\tilde{q}$ and $\tilde{r}$ are given in figure 1.

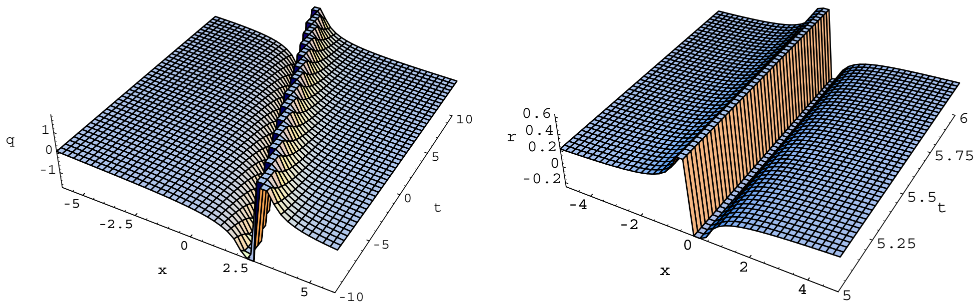

Figure 1. Soliton solutions $\tilde{q}$ and $\tilde{r}$ with $\lambda_1 = 0.01$, $\lambda_2 = -0.2$, $\gamma_1 = 0.1$, $\gamma_2 = 0.3$.

5. Conclusions

In this paper, we have constructed the *N*-fold Darboux transformation (3.10) for the nonlinear Dirac system (1.2) based on its Lax pairs. As an application, we construct an explicit solution (4.3) of the Dirac system. If the solution (4.3) is further taken as a new seed solution, we can make the Darboux transformation (3.10) once again and engender another new solution. This process can be done continually and yield a series of soliton solutions of the Dirac system in theory. The *N*-fold Darboux transformation (3.10) presented here has some merits. Firstly, the solution $(\tilde{q}, \tilde{r})$ in (3.10) is the *N*-fold Darboux transformation of the solution (q, r). It can be interpreted as a nonlinear superposition of the initial solution (q, r) and *N*-soliton solution. It contains all pure *N*-soliton solutions of the Dirac system (1.2) in a unified form. Therefore, it provides a unified and explicit *N*-soliton solutions for the Dirac system. Secondly, according to the Darboux transformation (3.10), the solutions of the Dirac system (1.2) are reduced to solving a linear algebraic system (3.5)–(3.6) which is easy to produce its multi-soliton solutions by symbolic computation on a computer.

Acknowledgments

The author would like to express his sincere thanks to the referees for their helpful suggestions. The work described in this paper was supported by grants from the National Science Foundation of China (10371023), the Shanghai Foundation of Educational Committee and the Shanghai Shuguang Project of China (02SG02).

References

[1] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[2] Gu C H, Hu H S and Zhou Z X 1999 *Darboux Transformation in Soliton Theory and its Geometric Applications* (Shanghai: Shanghai Scientific and Technical Publishers)
[3] Levi D 1988 *Inverse Problems* **4** 165
[4] Gu C H and Hu H S 1986 *Lett. Math. Phys.* **11** 325
[5] Gu C H and Zhou Z X 1994 *Lett. Math. Phys.* **32** 1
[6] Estevez P G 1999 *J. Math. Phys.* **40** 1406
[7] Neugebauer G and Meinel R 1984 *Phys. Lett.* A **100** 467
[8] Geng X G and Tam H W 1999 *J. Phys. Soc. Japan.* **68** 1508
[9] Frolov I S 1972 *Sov. Math. Dokl.* **13** 1468
[10] Grosse H 1986 *Phys. Rep.* **134** 297
[11] Hinton D B, Jordan A K, Klaus M and Shaw J K 1991 *J. Math. Phys.* **32** 3015
[12] Ma W X and Li K S 1996 *Inverse Problems* **12** L25
[13] Ma W X 1997 *Chin. Ann. Math.* B **18** 79
[14] Dickey L A 2003 *Soliton Equations and Hamiltonian Systems* 2nd edn (Singapore: World Scientific)